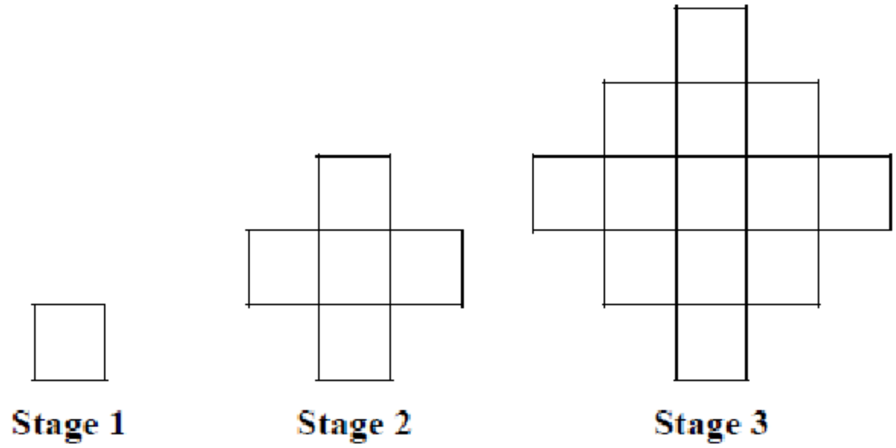
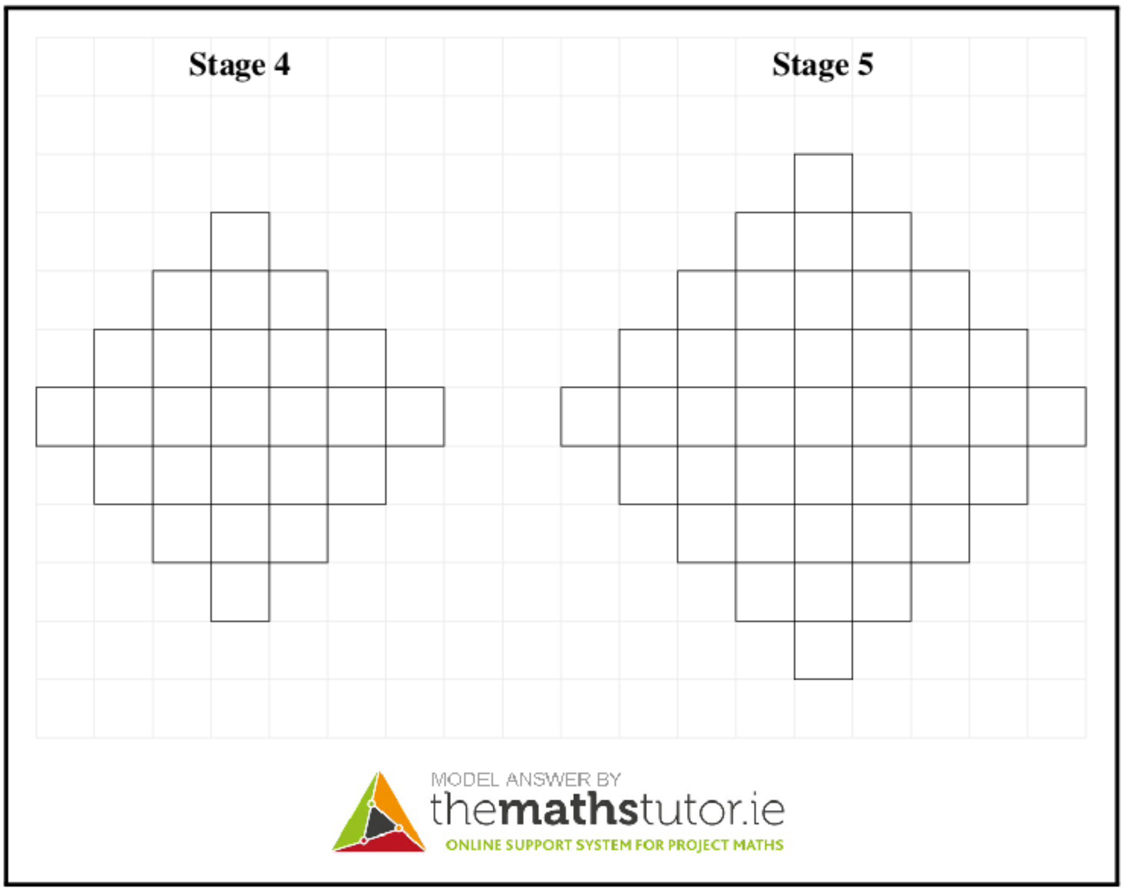
Question 1

**Question 8** **(Suggested maximum time: 20 minutes)**

The first three stages of a pattern are shown below. Each stage of the pattern is made up of small squares. Each small square has an area of one square unit.




**(a)** Draw the next two stages of the pattern.




**(b)** The perimeter of Stage 1 of the pattern is 4 units. The perimeter of Stage 2 of the pattern is 12 units. Find a general formula for the **perimeter** of Stage $n$ of the pattern, where $n \in \mathbb{N}$

We'll consider the perimeters of the first five Stages:

| **Stage** | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Perimeter** | 4 units | 12 units | 20 units | 28 units | 36 units |

From one stage to another, there is a difference of 8 in the perimeter. Thus, the perimeter is an arithmetic sequence with first element $a = 4$ and common difference $d = 8$. The general formula for the perimeter of Stage $n$ is therefore

$$P_n = a + (n-1)d = 4 + 8(n-1) = 8n - 4$$

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**(c)** Find a general formula for the **area** of Stage $n$ of the pattern, where $n \in \mathbb{N}$

We will test to see if the sequence is linear by considering the first differences:

$$d_1 = A_2 - A_1 = 5 - 1 = 4$$
$$d_2 = A_3 - A_2 = 13 - 5 = 8$$
$$d_3 = A_4 - A_3 = 25 - 13 = 12$$
$$d_4 = A_5 - A_4 = 41 - 25 = 16$$

The first differences are not constant, so the sequence is not arithmetic. We will look at the second differences:

$$s_1 = d_2 - d_1 = 8 - 4 = 4$$
$$s_2 = d_3 - d_2 = 12 - 8 = 4$$
$$s_3 = d_4 - d_3 = 16 - 12 = 4$$

The second differences are constant, so the sequence is quadratic, and so has the general formula $A_n = an^2 + bn + c$ where $a$, $b$ and $c$ need to be determined. For a quadratic sequence of this form, the constant second difference is equal to $2a$, so we have $2a = 4$, or $a = 2$. This means our sequence has the form $A_n = 2n^2 + bn + c$

Now, we know that $A_1 = 1$ and $A_2 = 5$, so

$$1 = A_1 = 2(1)^2 + b(1) + c = 2 + b + c \qquad \Rightarrow \qquad b + c = -1$$
$$5 = A_2 = 2(2)^2 + b(2) + c = 8 + 2b + c \qquad \Rightarrow \qquad 2b + c = -3$$

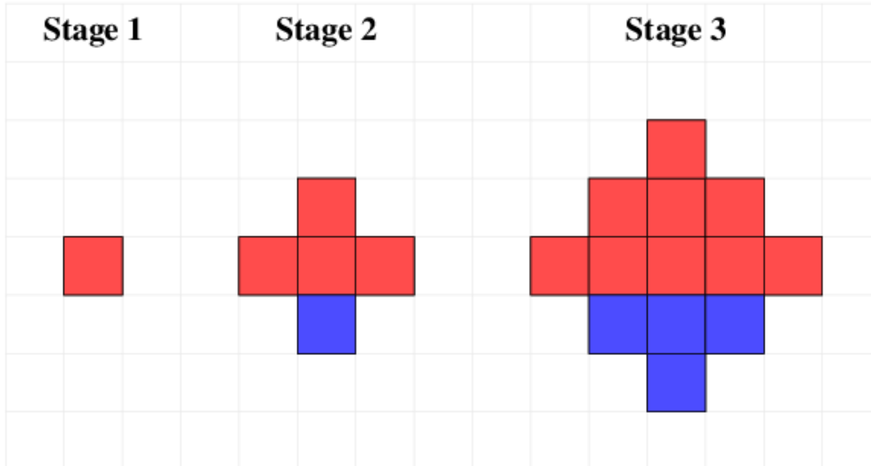
We will solve these equations simultaneously:

$$\begin{array}{rcl} b + c & = & -1 \\ 2b + c & = & -3 \\ \hline b & = & -2 \end{array}$$

Now, from the first of our two simultaneous equations to see that $(-2) + c = -1$ or $c = -1 + 2 = 1$. This means that our sequence has the form $A_n = 2n^2 - 2n + 1$

ALTERNATE SOLUTION

It will be easier to split the patterns into an upper and a lower section. The first three Stages are:




We will count the number of small squares in each section for the first five Stages

| **Stage** | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Upper** | 1 square | 4 squares | 9 squares | 16 squares | 25 squares |
| **Lower** | 0 squares | 1 squares | 4 squares | 9 squares | 16 squares |
| **Total Area** | 1 units$^2$ | 5 units$^2$ | 13 units$^2$ | 25 units$^2$ | 41 units$^2$ |

From this table, we can see that for stage $n$, there will be $n^2$ squares in the upper section and $(n-1)^2$ squares in the lower section. The general formula for the perimeter of Stage $n$ is therefore

$$A_n = n^2 + (n-1)^2 = n^2 + n^2 - 2n + 1 = 2n^2 - 2n + 1 \text{ units}^2$$

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**(d)** What kind of sequence (linear, quadratic, exponential, or none of these) do the areas follow? Justify your answer.

This sequence is quadratic because the highest power in the general formula is $n^2$. It cannot be exponential because there is no term of the form $a^n$.

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Question 2

**(i)** Complete the table.

| $p$ | $6p+1$ | $6p+5$ |
|---|---|---|
| 0 | **1** | **5** |
| 1 | 7 | 11 |
| 2 | 13 | 17 |
| 3 | 19 | 23 |
| 4 | 25 | 29 |
| 5 | 31 | 35 |

**(ii)** Give two reasons why his method is not fully correct.

*There are a number of different reasons – **any** two will suffice.*
*Reasons related to "**all** prime numbers":*

The formulas do not generate 2, which is prime.

The formulas do not generate 3, which is prime.

*Reasons related to "**only** prime numbers":*

The formulas generate 1, which is not prime.

The formulas generate 25, which is not prime.

The formulas generate 35, which is not prime.

**(b)** The Swiss mathematician and physicist, Euler, first noticed (in 1772) that the expression $n^2 - n + 41$ gives a prime number for all positive integer values of $n$ less than $41$.

Explain why it does not give a prime number for $n = 41$.

$41^2 - 41 + 41 = 41^2$, which has 41 as a factor.

Question 3

**(i)** Is the pattern of heights in the table linear, quadratic, or exponential? Explain your answer.

| **Time (seconds)** | 0 | 0·5 | 1 | 1·5 | 2 | 2·5 | 3 |
|---|---|---|---|---|---|---|---|
| **Height (metres)** | 0·3 | 3·4 | 5·7 | 7·2 | 7·9 | 7·8 | 6·9 |

*First difference*: 3·1 2·3 1·5 0·7 –0·1 –0·9

*Second difference*: –0·8 –0·8 –0·8 –0·8 –0·8

Answer: Quadratic.

Reason: The first differences are not all the same, but the second differences are.

**(ii)** Estimate the height of the ball after 3·5 seconds.

5·2 metres.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *Second difference*: | | | –0·8 | | –0·8 | | |
| *First difference*: | | –0·1 | | –0·9 | | –1·7 | |
| *Height (m)*: | 7·9 | | 7·8 | | 6·9 | | 5·2 |
| *Time (s)*: | 2 | | 2·5 | | 3 | | 3·5 |

**(iii)** Estimate the total time the ball spends in the air. Justify your answer.

*Continuing the method for* **(ii)**:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Second difference*: | | | –0·8 | | –0·8 | | –0·8 | | –0·8 | |
| *First difference*: | | –0·1 | | –0·9 | | –1·7 | | –2·5 | | –3·3 |
| *Height (m)*: | 7·9 | | 7·8 | | 6·9 | | 5·2 | | 2·7 | | –0·6 |
| *Time (s)*: | 2 | | 2·5 | | 3 | | 3·5 | | 4 | | 4·5 |

Answer: The ball spends roughly 4·4 seconds in the air. Its height is 0 just before 4·5 seconds.

*Or, graphically*:

From the graph, the ball spends roughly 4·4 seconds in the air